8.2 day 1
L’Hôpital’s Rule

Actually, L’Hôpital’s Rule was developed by his teacher Johann Bernoulli. De l’Hôpital paid Bernoulli for private lessons, and then published the first Calculus book based on those lessons.

Guillaume De l'Hôpital
1661 - 1704

Greg Kelly, Hanford High School, Richland, Washington

Consider: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

If we try to evaluate this by direct substitution, we get: $\frac{0}{0}$

Zero divided by zero can not be evaluated, and is an example of **indeterminate form**.

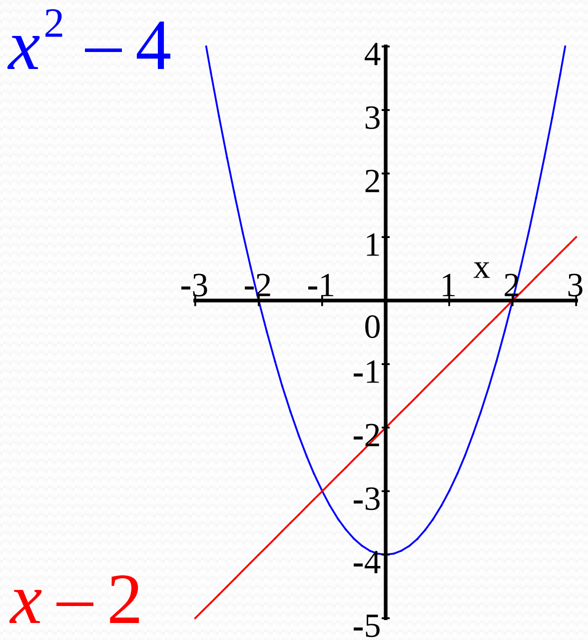
In this case, we can evaluate this limit by factoring and canceling:

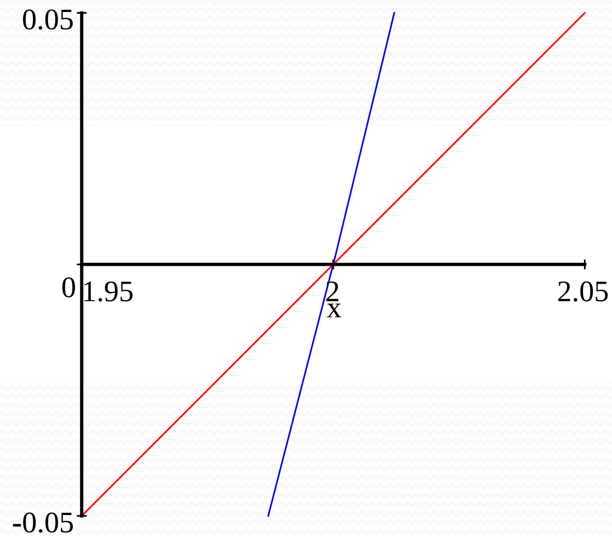
$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2} (x+2) = 4$$

→

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2-4}{x-2}$$

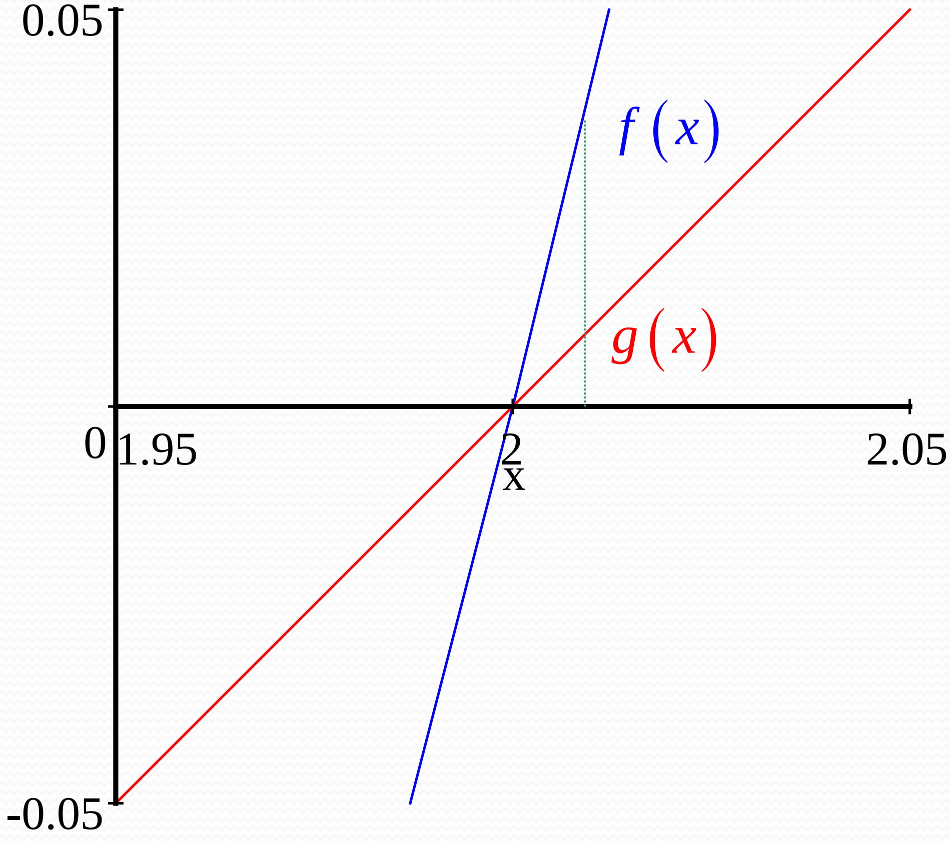
The limit is the ratio of the numerator over the denominator as x approaches 2.





→

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

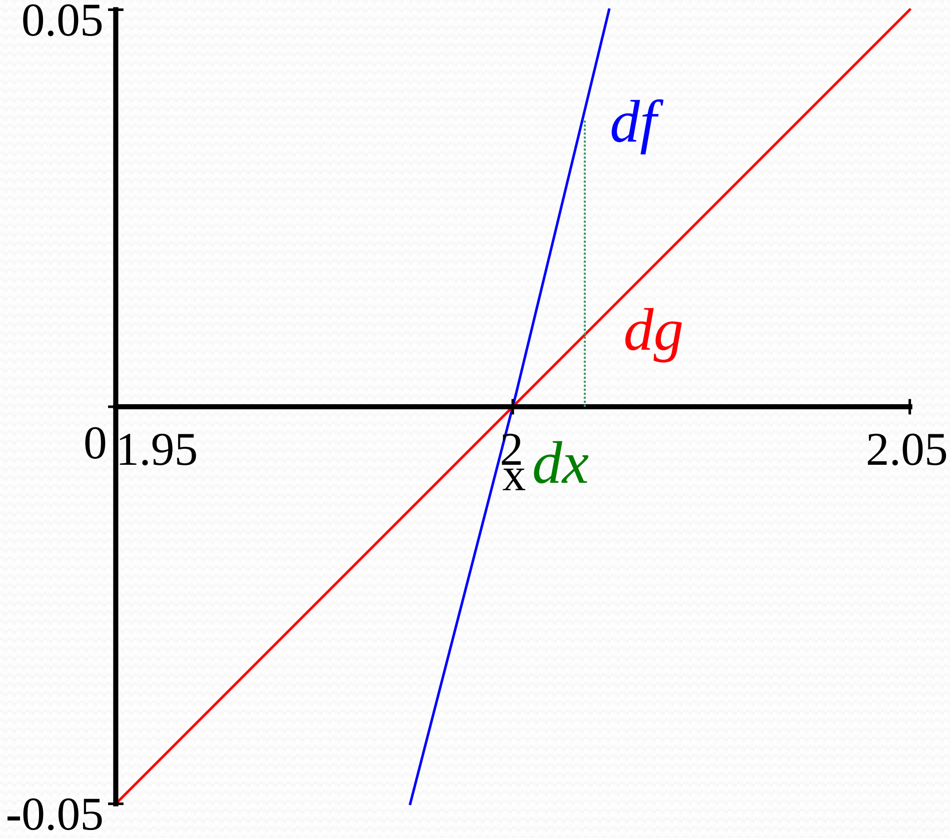


As $x \to 2$

$\frac{f(x)}{g(x)}$ becomes:

→

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2-4}{x-2}$$



As $x \to 2$

$\frac{f(x)}{g(x)}$ becomes:

$$\frac{df}{dg} = \frac{\frac{df}{dx}}{\frac{dg}{dx}}$$

→

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{\frac{d}{dx}\left(x^2-4\right)}{\frac{d}{dx}\left(x-2\right)} = \lim_{x\to 2}\frac{2x}{1} = 4$$

L'Hôpital's Rule:

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is indeterminate, then:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

We can confirm L'Hôpital's rule by working backwards, and using the definition of derivative:

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} \qquad = \lim_{x\to a}\frac{f(x)-0}{g(x)-0} \quad = \lim_{x\to a}\frac{f(x)}{g(x)}$$

→

Example:

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} = \lim_{x\to 0}\frac{\sin x}{1+2x} = 0$$

If it's no longer indeterminate, then **STOP**!

If we try to continue with L'Hôpital's rule:

$$= \lim_{x\to 0}\frac{\sin x}{1+2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$$

which is wrong, wrong, wrong!

→

On the other hand, you can apply L'Hôpital's rule as many times as necessary as long as the fraction is still indeterminate:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \longleftarrow \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1-\frac{1}{2}x}{x^2}$$

(Rewritten in exponential form.)

$$=\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \longleftarrow \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} \longleftarrow \text{not } \frac{0}{0}$$

$$=\frac{-\frac{1}{4}}{2}$$

$$=-\frac{1}{8}$$

→

L'Hôpital's rule can be used to evaluate other indeterminate forms besides $\frac{0}{0}$.

The following are also considered indeterminate:

$$\frac{\infty}{\infty} \qquad \infty \cdot 0 \qquad \infty - \infty \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

The first one, $\frac{\infty}{\infty}$, can be evaluated just like $\frac{0}{0}$.

<u>The others must be changed to fractions first</u>.

$\rightarrow$

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$$ ← This approaches $\infty \cdot 0$

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$$ ← This approaches $\frac{0}{0}$

We already know that $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$

but if we want to use L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\cos\left(\frac{1}{x}\right)=\cos\left(0\right)=1$$

→

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$ ← This is indeterminate form $\infty - \infty$

If we find a common denominator and subtract, we get:

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$ ← Now it is in the form $\frac{0}{0}$

$$\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}\right)$$ ← L'Hôpital's rule applied once.

$$\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$ ← Fractions cleared. Still $\frac{0}{0}$

→

$$\lim_{x \to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$$

$$\lim_{x \to 1}\left(\frac{1}{1+1+\ln x}\right)$$ ← L'Hôpital again.

$$\lim_{x \to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$

$$\frac{1}{2}$$

$$\lim_{x \to 1}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}\right)$$

$$\lim_{x \to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$

→

Indeterminate Forms: 1^{∞} 0^{0} ∞^{0}

Evaluating these forms requires a mathematical trick to change the expression into a fraction.

$$\ln u^{n} = n \ln u = \frac{\ln u}{\frac{1}{n}}$$

We can then write the expression as a fraction, which allows us to use L'Hôpital's rule.

$$\lim_{x \to a} f(x) = e^{\ln\left(\lim_{x \to a} f(x)\right)} = e^{\lim_{x \to a} \ln\left(f(x)\right)}$$

Then move the limit notation outside of the log.

We can take the log of the function as long as we exponentiate at the same time.

→

Indeterminate Forms: 1^{∞} 0^{0} ∞^{0}

Example:

$$\lim_{x\to\infty} x^{1/x} \leftarrow \infty^{0}$$

$$e^{\lim_{x\to\infty} \ln\left(x^{1/x}\right)}$$

$$e^{\lim_{x\to\infty} \frac{1}{x}\ln(x)}$$

$$e^{\lim_{x\to\infty} \frac{\ln(x)}{x}} \leftarrow \frac{\infty}{\infty}$$

$$e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}} \leftarrow$$ L'Hôpital applied

$$e^{0}$$

$$1$$

π